

# TRANSONIC ELASTIC MODEL FOR WIGGLY GOTO-NAMBU STRING

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October, 1994.

**Abstract.** *The hitherto controversial proposition that a “wiggly” Goto-Nambu cosmic string can be effectively represented by an elastic string model of exactly transonic type (with energy density  $U$  inversely proportional to its tension  $T$ ) is shown to have a firm mathematical basis.*

For many years after work on cosmic strings was initiated by Kibble[1] the subject was restricted to phenomena describable by a simple Goto-Nambu model representing a vacuum defect of the non-conducting kind whose prototype is the Nielsen-Olesen vortex[2]. More recently there has been a new wave of cosmological interest in “superconducting cosmic strings”, meaning phenomena due to a vacuum defect of the current carrying kind whose prototype is the Witten vortex[3]. It was suggested at an early stage[4] that a Witten vortex could be represented by an elegant current carrying model previously found by Nielsen[5] as the outcome of a Kaluza Klein type projection from an ordinary Goto-Nambu model in a background with an extra dimension. However it soon became clear that an adequate allowance even for weak currents requires the use of elastic string models of a more general category[6] that is characterised by *two* kinds of perturbation mode with generically distinct propagation speeds[7] namely an extrinsic “wiggly” (i.e. worldsheet displacement) perturbation speed  $c_E$  say and a sound type “woggle” (i.e. longitudinal) perturbation speed  $c_L$  say (of which the latter has no analogue in the internally structureless Goto-Nambu case for which the former is equal to the speed of light, set here to unity).

What disqualified the particular model obtained by the Nielsen mechanism[5] as an accurate macroscopic representation of a Witten vortex was the demonstration[8] that it is of *permanently transonic* type, meaning that it has the mathematically convenient but physically restrictive feature of equality of the “wiggly” and “woggle” propagation speeds, i.e.

$$c_E = c_L , \tag{1}$$

whereas Witten vortices were shown[9][10] to require representation by elastic string models of *supersonic* type, as characterised by the *strict* inequality  $c_E > c_L$ .

A positive by-product of the analysis[8] leading to this negative conclusion was the idea of an alternative application[11] for elastic string models of the permanently transonic kind characterised by (1). This was for the smoothed out macroscopic representation of an “ordinary” (Goto-Nambu) cosmic string with a (thermal or other) spectrum of “wiggles” that (for numerical or other reasons) one does not wish to describe in detail. The aim of the present letter is to settle the controversy that has arisen[12][13] about this proposal. Unlike the new constructive proof presented here, the original argument[11] was merely heuristic, while the confirmation[12] and its contradiction[13] did not deal directly with the evolution of the smoothed out world-sheet as is done here, but merely with the question of whether – for an underlying Goto-Nambu model whose energy density  $U$  and tension  $T$  have microscopically constant values  $U = T = m^2$  say – the corresponding variably “renormalised”[14][15] *average* values accurately satisfy the predicted[11] relation

$$UT = m^4 . \tag{2}$$

The objection legitimately raised[13] against the previous confirmation[12] was merely that it did not apply to quite the most general type of “wiggles”, but only to a restricted (transversely generated) class. However a more subtle issue in any such verification is that the prediction[11] concerned the *effective* energy density  $U$  and tension  $T$  of the smoothed out interpolating string model, whereas the litigious calculations[12][13] were concerned with *averages* over the “wiggles” in the original Goto-Nambu model. The delicate question is which of variously weighted averages  $\bar{\epsilon}$ ,  $\langle \epsilon \rangle$ , ... of the microscopic energy density  $\epsilon$  is to be identified with the “renormalised” effective value,  $U$ , and likewise for the tension. Since getting the “correct” answer depends on using the “correct” definition, the issue of how accurately (2) holds is to some extent semantic.

The new (constructive) derivation of the smoothed out transonic elastic string representation does not involve any such averaging procedure and thus bypasses the semantic issue implicit in the preceding debate[12][13]. The apparent discrepancy has moreover been recently resolved by Martin[16], who has shown that the predicted relation (2) is in fact obtained (even for non transversely generated wiggles) provided one uses the appropriate “effective” average as defined for energy by  $\bar{\epsilon} = E/x$  where  $E = \int \epsilon d\sigma$  is the total energy integrated along the length  $\sigma = \int d\sigma$  of a “wiggly” segment of the worldsheet, and  $x$  is the shortcut (background coordinate) distance between its ends, which is given, in terms of the cosine  $x' = dx/d\sigma$  of the angle

between the “wiggly” path and the shortcut interpolation, by  $x = \int x' d\sigma$ . In terms of the “path” averages defined by  $\langle \varepsilon \rangle = \sigma^{-1} \int \varepsilon d\sigma$  and  $\langle x' \rangle = \sigma^{-1} \int x' d\sigma$  the required “effective” average is  $\bar{\varepsilon} = \langle \varepsilon \rangle / \langle x' \rangle$ . The reported deviations[13] are attributable to the use of a slightly different “intermediate” average defined by  $\tilde{\varepsilon} = \langle \varepsilon / x' \rangle$ .

Showing how to fix the definitions so as to get a formula of the predicted form (2) is by itself not quite sufficient to show that the proposed model[11] can actually achieve its purpose of smoothly interpolating through the “wiggles” of the underlying Goto-Nambu string. To do this, one must go back to the equations of motion which, both for a Goto-Nambu string and for strings of the perfectly elastic category[6] needed here (though not for more general multiply conducting models[17]) are expressible completely (so long as there are no self intersections) just by the standard force balance relation[8]

$$\bar{\nabla}_\mu \bar{T}^{\mu\nu} = \bar{f}^\nu , \quad (3)$$

where  $\bar{\nabla}_\mu$  denotes the worldsheet projected covariant differentiation operator whose definition is recapitulated below, while  $\bar{f}^\mu$  is the external electromagnetic or other (e.g. material drag[18]) force density, which will be assumed here to vanish, and  $\bar{T}^{\mu\nu}$  is the surface stress momentum energy density of the string. The latter, which is tensorial with respect to the spacetime coordinates  $x^\mu$  ( $\mu=0,1,2,3,4$ ), is given in terms of its internal analogue  $t^{ab}$  which is tensorial with respect to internal worldsheet coordinates  $\sigma^a$  ( $a=0,1$ ) by

$$\bar{T}^{\mu\nu} = t^{ab} x_{,a}^\mu x_{,b}^\nu , \quad t^{ab} = 2 \frac{\partial \mathcal{L}}{\partial h^{ab}} + \mathcal{L} h^{ab} , \quad (4)$$

(using a comma for partial differentiation) where  $\mathcal{L}$  is the surface Lagrangian density (which is just a constant in the Goto-Nambu case) and  $h^{ab}$  is the contravariant version of the induced metric, whose spacetime projection gives the fundamental tangential projection tensor

$$\eta^{\mu\nu} = h^{ab} x_{,a}^\mu x_{,b}^\nu , \quad h_{ab} = g_{\mu\nu} x_{,a}^\mu x_{,b}^\nu , \quad (5)$$

that is needed for defining the tangentially projected derivation operator in (3) and also the orthogonal projection tensor  $\perp_\nu^\mu$  that will be needed below:

$$\bar{\nabla}_\mu = \eta_\mu^\nu \nabla_\nu , \quad \perp_\nu^\mu = g_\nu^\mu - \eta_\nu^\mu . \quad (6)$$

The corresponding Dirac distributional stress momentum energy density tensor  $\hat{T}^{\mu\nu}$  say over the background is given by

$$\hat{T}^{\mu\nu} = \|g\|^{-1/2} \int \bar{T}^{\mu\nu} \delta^4[x - x\{\sigma\}] \sqrt{\|h\|} d^2\sigma . \quad (7)$$

It is easily verified that vanishing of the force term in the regular differential equation (3) is equivalent to conservation (in the usual sense) of this distribution:

$$\bar{f}^\mu = 0 \quad \Leftrightarrow \quad \nabla_\mu \hat{T}^{\mu\nu} = 0 . \quad (8)$$

In a generic state,  $\bar{T}^{\mu\nu}$  will have timelike and spacelike unit worldsheet tangential eigenvectors  $u^\mu$  and  $v^\mu$  say that define a preferred orthonormal tangent frame in terms of which one obtains

$$\bar{T}^{\mu\nu} = U u^\mu u^\nu - T v^\mu v^\nu , \quad \eta^{\mu\nu} = -u^\mu u^\nu + v^\mu v^\nu , \quad (9)$$

in which the eigenvalues are the energy density  $U$  and tension  $T$  referred to above. The permanently transonic model proposed for representing the smoothed interpolating worldsheet is obtainable, in accordance with (4) from a Lagrangian density of the form  $\mathcal{L} = -m^2 \sqrt{1 + h^{ab} \psi_{,a} \psi_{,b}}$  where  $\psi$  is an independently variable stream function (whose gradient gives the direction of the spacelike eigenvector  $v^\mu$ ) which leads to  $U = -\mathcal{L}$  and  $T = -m^4/\mathcal{L}$ , in evident accord with (2). The underlying Goto Nambu model that one wants to represent this way is given more simply by  $\mathcal{L} = -m^2$  which gives  $U = T = m^2$  while leaving the eigendirections unspecified. In both the Goto Nambu and the generic elastic cases, (9) can be conveniently rewritten in the form

$$\bar{T}^{\mu\nu} = \frac{1}{2} (\beta_+^\mu \beta_-^\nu + \beta_-^\mu \beta_+^\nu) , \quad \beta_\pm^\mu = \sqrt{U} u^\mu \pm \sqrt{T} v^\mu . \quad (10)$$

Taking the orthogonal projection (by contraction with  $\perp_\nu^\mu$ ) of the dynamical relations (3) and combining the result with the kinematic identity

$$\perp_\nu^\mu (\beta_+^\rho \nabla_\rho \beta_-^\nu - \beta_-^\rho \nabla_\rho \beta_+^\nu) = 0 , \quad (11)$$

(which holds just as Frobenius type integrability condition for the vectors  $\beta_\pm^\nu$  to be and to remain worldsheet tangential) one obtains the extrinsic part of the dynamical equations (3) in the neat characteristic form

$$\perp_\nu^\mu \beta_\pm^\rho \nabla_\rho \beta_\mp^\nu = \perp_\nu^\mu \bar{f}^\nu , \quad (12)$$

which is valid for any kind of elastic string model, and from which it can be seen that  $\beta_\mp^\nu$  are actually bicharacteristic and hence by their construction (10) that relative to the preferred frame specified by  $u^\mu$  the propagation speed for extrinsic “wiggle” perturbations will be given by the generally valid formula  $c_E = \sqrt{T/U}$ . When the force density  $\bar{f}^\nu$  is set to zero, it can be seen that (12) is expressible in terms of arbitrarily rescaled bicharacteristic vectors  $\ell_\mp^\mu = \alpha \beta_\mp^\mu$  simply as

$$\perp_\nu^\mu \ell_\pm^\rho \nabla_\rho \ell_\mp^\nu = 0 . \quad (13)$$

In the familiar Goto-Nambu case for which the bicharacteristic vectors are null there remains no tangentially contracted part of the equations of motion, which are completely contained in (12). Thus there is an indeterminacy which can be resolved, taking advantage of the rescaling freedom in (13), by taking the bicharacteristic vectors to be Lie transported along each other. The orthogonal projection in (12) can thereby be removed, leaving the equations of motion in the fully determinate form

$$\ell_{\pm}^{\rho} \nabla_{\rho} \ell_{\mp}^{\nu} = 0 . \quad (14)$$

This system is well known to be completely soluble in a flat background in terms of characteristic coordinates  $\sigma^{+}$  and  $\sigma^{-}$  such that

$$\ell_{\pm}^{\rho} = \frac{\partial x^{\mu}}{\partial \sigma^{\pm}} , \quad (15)$$

by the familiar ansatz[19] (on which the calculations referred to above[12][13] were based) given, using Minkowski coordinates  $x^{\mu}$ , by

$$x^{\mu} \{ \sigma^{+}, \sigma^{-} \} = x_{+}^{\mu} \{ \sigma^{+} \} + x_{-}^{\mu} \{ \sigma^{-} \} \quad (16)$$

for a pair of generating curves  $x_{\pm}^{\mu} \{ \tau \}$  that can “wiggle” freely subject to the restriction that their tangent vectors  $\dot{x}_{\pm}^{\mu} = dx_{\pm}^{\mu}/d\tau$  remain everywhere null, i.e.  $\dot{x}_{+}^{\mu} \dot{x}_{+\mu} = 0$  and  $\dot{x}_{-}^{\mu} \dot{x}_{-\mu} = 0$ . Thus if the parameter  $\tau$  is chosen to be just the Minkowski time coordinate  $x^0$ , so that  $\dot{x}_{\pm}^0 = 1$ , the space vectors with components  $\dot{x}_{\pm}^i$  for  $i = 1, 2, 3$  will lie on the surface of the (Kibble Turok) unit sphere.

To deal with the generic elastic case in which  $T$  is less than  $U$  so that the bicharacteristic vectors  $\beta_{\pm}^{\rho}$  will be not null but timelike, it is convenient to use a rescaling given by

$$\beta_{\pm}^{\rho} = \sqrt{U - T} \ell_{\pm}^{\rho} , \quad (17)$$

so as to obtain a pair of bicharacteristic vectors  $\ell_{\pm}^{\rho}$  that have unit normalisation. In this generic case the internal dynamical equations obtained by tangential projection of (3) are no longer trivial, but, in the special case (2) considered here, the pair of internal equations got by contracting (3) with the independent bicharacteristic vectors can be cast into the particularly simple form

$$\overline{\nabla}_{\mu} \left( (U - T) \ell_{\mp}^{\mu} \right) = -\ell_{\pm}^{\nu} \overline{f}_{\nu} . \quad (18)$$

By a more elaborate manipulation, these simple surface divergence equations replaced by what for the present purpose is the more useful combined equation

$$(U - T) \eta^{\mu}_{\nu} \ell_{\pm}^{\rho} \nabla_{\rho} \ell_{\mp}^{\nu} = (\eta^{\mu\nu} + \ell_{\mp}^{\mu} \ell_{\mp}^{\nu}) \overline{f}_{\nu} . \quad (19)$$

By showing that the tangent vectors  $\ell_{\pm}^{\mu}$  are bicharacteristic for internal “woggles” not just for extrinsic “wiggles”, this demonstrates the transonicity property (1).

It is obvious by (6) that, when the force term on the right is absent, recombining the intrinsic dynamical equation (19) with its extrinsic analogue (13) reconstitutes the complete (unprojected) set of dynamical equations in *the same* simple form (14) as was obtained by judicious use of the gauge freedom for the null characteristic vectors of the Goto-Nambu case. This crucial result implies, as before, that in a Minkowski background the general solution will be obtainable in terms of characteristic coordinates  $\sigma^+$  and  $\sigma^-$  (corresponding to preferred internal time and space coordinates  $\tau = \sigma^+ + \sigma^-$ ,  $\sigma = \sigma^+ - \sigma^-$ ) by an ansatz of the same form (16) as in the Goto-Nambu case, the only difference being that, instead of being null, the separate generating curves are now required to be *timelike*. In order for the characteristic vectors given by (15) as  $\ell_{\pm}^{\mu} = \dot{x}_{\pm}^{\mu}$  to satisfy the unit normalisation condition, the separate generating curves would also need to be restricted to have a *proper time* parametrisation, but it is evident that this is not obligatory in order for (16) to be a solution. An alternative is the *standard* parametrisation given by the background time coordinate  $x^0$  on each generating curve, which (with  $\tau = x^0$ ) gives  $\dot{x}_{\pm}^0 = 1$ : for the generators to be timelike, the 3-vectors  $\dot{x}_{\pm}^i$  must then lie, not on the surface as in the Goto-Nambu case, but *inside* the (Kibble Turok) unit sphere.

To see how the solution that has just been given for the worldsheet of the transonic elastic model can be used to provide a smoothed interpolation through the “wiggles” of the worldsheet of an underlying Goto-Nambu model, it now suffices to use an idea introduced by Smith and Vilenkin[20] for the purpose of numerical computation, for which one needs to replace the exact continuous description of the worldsheet by a discrete representation. The Smith Vilenkin method is simply to use a pair of discrete sets of sampling points  $x_{\pm r}^{\mu} = x_{\pm}^{\mu}\{\sigma_r\}$  determined by a corresponding discrete set of parameter values  $\sigma_r$  on the generating curves of the exact representation (16). This provides a “diamond lattice” of sample points given (for integral values of r and s) by

$$x_{rs}^{\mu} = \frac{1}{2}(x_{+r}^{\mu} + x_{-s}^{\mu}) , \quad (20)$$

that will automatically lie *exactly* on the “wiggly” Goto-Nambu worldsheet (16), which is thus represented to any desired accuracy by choosing a sufficiently dense set of sampling parameter values  $\sigma_r$  on the separate “wiggly” null generating curves  $x_{\pm}^{\mu}\{\sigma\}$ . To construct a corresponding smoothed out worldsheet it suffices to consider the chosen set of sample points  $x_{\pm r}^{\mu} = x_{\pm}^{\mu}\{\sigma_r\}$  on the separate “wiggly” null generators to be sample points on a pair of *smoothed out* – and thus no longer null

but timelike – *interpolating curves*. Using these *timelike* interpolating curves as generators, the *same* ansatz (16) can now be used to construct another smoother *interpolating worldsheet* that will satisfy (14) and therefore will be *an exact solution* of the dynamic equations for the transonic elastic string model. (The form (16) of the general solution also shows incidentally that a similarly smoothed out interpolation for the worldsheet of a “wiggly” and “woggly” elastic string of transonic type will be provided by an effective model of *the same* transonic type.)

The (less “wiggly”) elastic string worldsheet given by this construction will obviously be an even better approximation to the (more “wiggly”) Goto-Nambu worldsheet than the original Smith Vilenkin lattice representation, which itself could already be made as accurate as desired by choosing a sufficiently high sampling resolution. No matter how far it is extrapolated to the future, the smoothed elastic string worldsheet can never stray significantly from the underlying “wiggly” Goto-Nambu worldsheet it is designed to represent (at least in the absence of background curvature whose effect remains a topic for future investigation) because the exact worldsheet and the elastic interpolation will always coincide precisely at each point of their shared Smith Vilenkin lattice (20). This highly satisfactory feature of guaranteed error cancellation in the long run could not be improved, but would only be spoiled, by any “deviation” from (2).

The arbitrarily accurate worldsheet matching property that has just been demonstrated, does not depend on the mass scale  $m$  in the action for the transonic elastic model being the same as that of the underlying Goto-Nambu model, because for both kinds of model the actual equations of motion are *scale independent*. The same mass scale is however needed if one wants agreement of the effective energy density and tension of the smoothed elastic model with corresponding (suitably defined) averages over the “wiggles” in the underlying Goto-Nambu model.

Having thus provided a solid (directly constructive) basis for the claim that the permanently transonic elastic string model provides an excellent description of the effect of microscopic wiggles in a Goto-Nambu string so long as self intersections remain unimportant (as was assumed throughout the debate[12][13] discussed above), it must be emphasised that the neglect of such intersections will not be justified when the effective temperature[11][17] of the “wiggles” is too high (as will presumably be the case[21] during a transient period immediately following the string - forming phase transition). Such intersections will produce microscopic loops, of which some will escape. Estimation of the dissipative force density  $\bar{f}^\nu$  that would be needed to allow for the “cooling” effect of such losses remains a problem for future work.

The author wishes to thank A. Vilenkin and X. Martin for helpful conversations.

## References.

- [1] T.W.B. Kibble, *J.Phys.* **A9**, 1387 (1976).
- [2] H.B. Nielsen, P. Olesen, *Nucl. Phys.* **B61**, 45 (1973).
- [3] E. Witten, *Nucl. Phys.***B249**, 557 (1985).
- [4] N.K. Nielsen, P. Olesen, *Nucl. Phys.* **B291**, 829 (1987).
- [5] N.K. Nielsen, *Nucl. Phys.* **B167** 248 (1980).
- [6] B. Carter, *Phys. Lett.* **B224** , 61 (1989).
- [7] B. Carter, *Phys. Lett.* **B228**, 446 (1989).
- [8] B. Carter, in *Formation and Evolution of Cosmic Strings*, ed. G. Gibbons, S. Hawking, T. Vachaspati, pp143-178 (Cambridge U.P., 1990).
- [9] P. Peter, *Phys. Rev.* **D45**, 1091 (1992).
- [10] P. Peter, *Phys. Rev.* **D47**, 3169 (1993).
- [11] B. Carter, *Phys. Rev.* **D41**, 3869 (1990).
- [12] A. Vilenkin, *Phys. Rev.* **D41**, 3038 (1990).
- [13] J.Hong, J.Kim, P. Sikivie, *Phys. Rev. Lett.* **69**, 2611 (1980).
- [14] B. Allen, E.P.S. Shellard, *Phys. Rev. Lett.* **64**, 119 (1990).
- [15] E.P.S. Shellard, B. Allen, in *Formation and Evolution of Cosmic Strings*, ed. G. Gibbons, S. Hawking, T. Vachaspati, pp421-448 (Cambridge U.P., 1990).
- [16] X. Martin, preprint (Observatoire de Paris, Meudon, 1994).
- [17] B. Carter, *Nucl. Phys.* **B412**, 345 (1994).
- [18] A. Vilenkin, *Phys. Rev.* **D43**, 1060 (1991).
- [19] T.W.B. Kibble, N. Turok, *Phys. Lett* **B116** 141 (1982).
- [20] A.G. Smith, A. Vilenkin, *Phys. Rev.* **D36** 990 (1987).
- [21] B. Carter, M. Sakellariadou, X. Martin, *Phys. Rev.* **D50**, 628 (1994).